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LETTER TO THE EDITOR

Exact multimagnon states in one-dimensional ferromagnetic spin chains with a short-range interaction

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Abstract. We propose the procedure of constructing the exact wavefunctions of M magnons on an infinite one-dimensional lattice in the one-parameter model with an exponentially decreasing interaction between spins. For $M \leqslant 4$ the problem is solved explicitly. The Bethe wavefunctions in the Heisenberg XXX chain appear in the limit of nearest-neighbour interaction.

The investigation of quantum spin chains with nearest-neighbour interaction has proven to be extremely fruitful in studies of exactly solvable models in statistical mechanics [1]. The common feature of almost all these models is the solution in the form of the Bethe ansatz [2] proposed sixty years ago for the $S=\frac{1}{2}$ isotropic Heisenberg chain. After the Bethe paper, however, a very long time passed. The next candidates for the role of integrable $S=\frac{1}{2}$ isotropic models with exchange integrals depending only on the distance r between spins have been proposed only recently [3, 4, 7]. The interaction in these chains has not been restricted to nearest-neighbour spins and, contrary to the Bethe solution, the description of all possible states has not been given. For the long-range r^{-2} exchange under periodic boundary conditions Haldane [3] and Shastry [4] have found only some sets of Jastrow-type wavefunctions. The simplest two-magnon states and some integrals of motion have been described for a more complicated model with short-range interaction [7] which for an infinite chain is determined by the Hamiltonian

$$H=\frac{J_0}{2}\sum_{j,k=-\infty}^{\infty} h(j-k)\frac{(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k-\mathbb{1})}{2} \qquad h(j)=\left[\frac{\kappa}{\pi}\sinh\frac{\pi}{\kappa}j\right]^{-2}. \tag{1}$$

Here $\{\boldsymbol{\sigma}_j\}$ are the usual Pauli matrices and κ is an arbitrary positive number.

This model contains both the Heisenberg and Haldane–Shastry chains as limits as $\kappa\to 0, \infty$ under proper renormalization of the coupling J_0. The periodic version of (1) with N interacting spins can also be obtained by replacing the trigonometric exchange integrals $h(j-k)$ with the Weierstrass elliptic functions $\wp(j-k)$ [6] with the real period N [7]. Due to the complexity of the Hamiltonian at this stage of development, neither any regular way of constructing all states of these systems nor appropriate use of standard criteria [5] for the proof of integrability have been indicated.

In this letter we report on an extension of the Bethe ansatz which allows one to obtain the exact wavefunctions of M magnons with arbitrary quasimomenta in the ferromagnetic case of the model (1) $J_0<0$. The result for $M\leqslant 4$ is presented in an

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explicit form. To our knowledge, it is the first case of an exact description of more than two interacting magnons in an infinite $S=\frac{1}{2}$ isotropic chain with exchange $h(j-k)$ since the Bethe paper [2].

The diagonalization of the Hamiltonian (1) can be reduced to finding completely symmetric tensors $\psi_{\{n_\alpha\}}(1\leqslant\alpha\leqslant M,\ n_\alpha\in\mathbb{Z})$ obeying the equation

$$\mathscr{L}\psi_{\{n_\alpha\}}=\sum_{\beta=1}^{M}\sum_{p\neq\{n_\alpha\}}^{\infty}h(n_\beta-p)\psi_{\{n_\alpha\}}^{(p,n_\beta)}+\psi_{\{n_\alpha\}}\left(\sum_{\beta\neq\gamma}^{M}h(n_\beta-n_\gamma)-J_0^{-1}\varepsilon^{(M)}-M\varepsilon_0\right)=0 \qquad (2)$$

where $\varepsilon_0=\Sigma_{j\neq0}\,h(j)$ and $\psi_{\{n_\alpha\}}^{(p,n_\beta)}$ is the tensor obtained from $\psi_{\{n_\alpha\}}$ by replacing the βth index n_β with p. The eigenvectors of (1) have the form $|\psi_M\rangle=\Sigma_{\{n_\alpha\}}\,\psi_{\{n_\alpha\}}|\{n_\alpha\}\rangle$, where in the states $|\{n_\alpha\}\rangle$ spins at positions $\{n_\alpha\}$ are turned over the ferromagnetic ground state. So $\psi_{\{n_\alpha\}}$ should vanish for any pair of coinciding $\{n_\alpha\}$. In the case $M=2$, the solution to (2) given in [7] can be written in the compact Bethe-like form

$$\psi_{n_1n_2}=\sum_{P\in\pi_2}\exp\left(\mathrm{i}\sum_{\alpha=1}^{2}k_{P\alpha}n_\alpha\right)\frac{S_{12}^{(P)}}{s_{12}}. \qquad (3)$$

Here $\{k_\alpha\}$ are magnon quasimomenta ($|\mathrm{Im}\ k_\alpha|<2\pi\kappa^{-1}$), π_M is the group of all permutations P of M indices and

$$S_{\alpha\beta}^{(P)}=\sinh\left[\frac{\pi}{\kappa}(n_\alpha-n_\beta)+\gamma(k_{P\alpha},k_{P\beta})\right] \qquad s_{\alpha\beta}=\sinh\frac{\pi}{\kappa}(n_\alpha-n_\beta). \qquad (4)$$

The two-magnon energy is $\varepsilon^{(2)}=J_0[\varepsilon(k_1)+\varepsilon(k_2)]$ with

$$\varepsilon(k)=-\left\{\mathscr{P}(r_k)+2\left[\zeta(r_k)-\frac{2r_k}{\omega}\zeta\left(\frac{\omega}{2}\right)\right]\left[\zeta(2r_k)-\zeta(r_k)-\frac{2r_k}{\omega}\zeta\left(\frac{\omega}{2}\right)\right]+\frac{2}{\omega}\zeta\left(\frac{\omega}{2}\right)\right\} \qquad (5)$$

where $\omega=\mathrm{i}\kappa$, $r_k=-k\omega(4\pi)^{-1}$ and $\mathscr{P}(x)$, $\zeta(x)$ are the usual Weierstrass elliptic functions with the periods l and ω. The phase shift in (4) depends on the quasimomenta as

$$\coth\gamma(k_1,k_2)=\frac{\kappa}{2\pi}[f(k_1)-f(k_2)] \qquad f(k)=\frac{k}{\pi}\zeta\left(\frac{\omega}{2}\right)-\zeta\left(\frac{k\omega}{2\pi}\right). \qquad (6)$$

At $M\geqslant2$ one can try to find the solution of (2) in the form of the Bethe ansatz, i.e. the symmetrized product of two-magnon amplitudes,

$$\psi_{\{n_\alpha\}}^{(0)}=\sum_{P\in\pi_M}\Phi^{(0)}(P;\{n_\alpha\})$$
$$\Phi^{(0)}(P;\{n_\alpha\})=\exp\left(\mathrm{i}\sum_{\alpha=1}^{M}k_{P\alpha}n_\alpha\right)\prod_{\mu>\nu}^{M}\frac{S_{\mu\nu}^{(P)}}{S_{\mu\nu}}. \qquad (7)$$

The calculation of the left-hand side of (2) can be performed by the theory of elliptic functions. It is sufficient to find the infinite sum

$$S(k,\{l_\alpha\},\{\Delta_\beta\})=\sum_{p\neq0,\{-l_\alpha\}}^{\infty}\exp(\mathrm{i}kp)R(p,\{l_\alpha\},\{\Delta_\beta\}) \qquad (8)$$

where

$$R(p,\{l_\alpha\},\{\Delta_\beta\})=\left[\frac{\kappa}{\pi}\sinh\frac{\pi}{\kappa}p\right]^{-2}\prod_{\alpha=1}^{K}\left[\sinh\frac{\pi}{\kappa}(p+l_\alpha)\right]^{-1}\prod_{\beta=1}^{K-2J}\sinh\left[\frac{\pi}{\kappa}(p+l_\beta)+\Delta_\beta\right]$$

K, J, $\{l_\alpha\}$ being $K+2$ integers such that $K-2J\geqslant 0$, $\Pi^K_{\beta\neq\gamma}(l_\beta-l_\gamma)\neq 0$, and $K-2J+1$ complex numbers $\{\Delta\}$, k are restricted to the strips $|\mathrm{Im}\,\Delta_\beta|<2\pi$, $|\mathrm{Im}\,k|<2\pi/\kappa(1+J)$. For that purpose one can consider the function

$$F(x)=\sum_{p=-\infty}^{\infty}\exp(\mathrm{i}kp)R(p+x,\{l_\alpha\},\{\Delta_\beta\})$$

which is doubly quasiperiodic: $F(x+1)=\exp(-\mathrm{i}k)F(x)$, $F(x+\omega)=F(x)$. It has the only pole singularity at $x=0$ on the torus $\mathbb{C}/\Gamma$ obtained by factorization of the complex plane $\mathbb{C}$ by the lattice of quasiperiods $\Gamma=m_1+m_2\omega$, $m_1, m_2\in\mathbb{Z}$. So, $F(x)$ can be constructed, as was done in [7] for more simple cases, from Weierstrass functions with periods l and ω. The sum (8) appears in the third term of the expansion of $F(x)$ near $x=0$, which allows one to obtain it in the closed form,

$$\begin{aligned}S(k,\{l_\alpha\},\{\Delta_\beta\})&\\ =\prod_{\alpha=1}^{K}\left[\sinh\frac{\pi}{\kappa}l_\alpha\right]^{-1}&\prod_{\beta=1}^{K-2J}\sinh\left(\frac{\pi}{\kappa}l_\beta+\Delta_\beta\right)\left\{\varepsilon(k)+\varepsilon_0+\frac{\pi}{\kappa}f(k)\varphi_{JK}-\frac{\pi^2}{2\kappa^2}\right.\\ &\times\left.\left[\varphi_{JK}^2+\sum_{\lambda=1}^{K}\left(\sinh\frac{\pi l_\lambda}{\kappa}\right)^{-2}-\sum_{\lambda=1}^{K-2J}\left[\sinh\left(\frac{\pi l_\lambda}{\kappa}+\Delta_\lambda\right)\right]^{-2}\right]\right\}\\ &+\frac{\pi}{\kappa}f(k)\left(\sum_{\lambda=1}^{K-2J}\eta_\lambda\sinh\Delta_\lambda+\sum_{\lambda=k-2J+1}^{K}\eta_\lambda\right)\\ &-\frac{\pi^2}{\kappa^2}\left(\sum_{\lambda=1}^{K-2J}\eta_\lambda\sinh\Delta_\lambda(\coth\Delta_\lambda+\xi_\lambda)+\sum_{\lambda=K-2J+1}^{K}\eta_\lambda\xi_\lambda\right)\end{aligned}\tag{9}$$

where $\varepsilon(k)$ and $f(k)$ are as in (5), (6), and

$$\varphi_{JK}=\sum_{\lambda=1}^{K-2J}\coth\left(\frac{\pi}{\kappa}l_\lambda+\Delta_\lambda\right)-\sum_{\lambda=1}^{K}\coth\frac{\pi l_\lambda}{\kappa}$$

$$\eta_\lambda=\exp(-\mathrm{i}kl_\lambda)\left[\sinh\frac{\pi l_\lambda}{\kappa}\right]^{-2}\prod_{2\nu\neq\lambda}^{K-2J}\sinh\left[\frac{\pi}{\kappa}(l_\nu-l_\lambda)+\Delta_\nu\right]\prod_{\rho\neq\lambda}^{K}\left[\sinh\frac{\pi}{\kappa}(l_\rho-l_\lambda)\right]^{-1}$$

$$\xi_\lambda=2\coth\frac{\pi l_\lambda}{\kappa}+\sum_{\nu\neq\lambda}^{K-2J}\coth\left[\frac{\pi}{\kappa}(l_\nu-l_\lambda)+\Delta_\nu\right]\sum_{\nu\neq\lambda}^{K}\coth\frac{\pi}{\kappa}(l_\nu-l_\lambda).$$

The very long and complicated expression for $\mathscr{L}\psi^{(0)}_{\{n_\alpha\}}$, which is obtained by the use of (9) at $J=0$, simplifies essentially if the identity for two-magnon phase shifts in our model

$$\coth\gamma(k_1,k_2)+\coth\gamma(k_2,k_3)+\coth\gamma(k_3,k_2)=0\tag{10}$$

is taken into account. We find that, contrary to the Heisenberg case, the Bethe ansatz (7) is not a solution of (2). If the energy of M magnons is chosen as $\varepsilon^{(M)}=J_0\sum_{\alpha=1}^{M}\varepsilon(k_\alpha)$, the left-hand side of (2) has the form

$$\begin{aligned}\mathscr{L}\psi^{(0)}_{\{n_\alpha\}}=\frac{\pi^2}{\kappa^2}\sum_{P\in\pi_M}\Phi^{(0)}(P;\{n_\alpha\})&\sum_{\mu\neq\nu\neq\rho}\frac{C^{(P)}_{\mu\nu}G^{(P)}_{\mu\rho}C^{(P)}_{\nu\rho}}{S^{(P)}_{\mu\nu}S^{(P)}_{\mu\rho}S^{(P)}_{\nu\rho}}\\ \times\Big\{\tfrac{1}{6}(2+d_{\mu\nu}D^{(P)}_{\mu\nu}&+d_{\mu\rho}D^{(P)}_{\mu\rho}+d_{\nu\rho}D^{(P)}_{\nu\rho})-\tfrac{1}{2}\exp[\mathrm{i}k_{P\mu}(n_\nu-n_\mu)]D^{(P)}_{\mu\nu}\\ \times(d_{\mu\nu}+d_{\nu\rho})&\prod_{\lambda\neq\mu,\nu,\rho}^{M}\sinh\left[\frac{\pi}{\kappa}(n_\nu-n_\lambda)+\gamma(k_{P\mu},k_{P\lambda})\right]s_{\mu\lambda}(s_{\nu\lambda}S^{(P)}_{\mu\lambda})^{-1}\Big\}\end{aligned}\tag{11}$$

where $s_{\mu\lambda}$, $S_{\mu\lambda}^{(P)}$ are given by (4) and $d_{\mu\nu}=\coth(\pi/\kappa)(n_\mu-n_\nu)$, $C_{\mu\nu}^{(P)}=\sinh\gamma(k_{P_\mu},k_{P_\nu})$, $D_{\mu\nu}^{(P)}=\coth\gamma(k_{P_\mu},k_{P_\nu})$. Due to the factors of the type $C_{\mu\nu}^{(P)}[S_{\mu\nu}^{(P)}]^{-1}$, (11) falls off exponentially with the increase of the distances $|n_\alpha-n_\beta|$ between all turned spins.

So (7) looks like a good approximation to the genuine solution. Motivated by the structure of (11), we propose for the exact wavefunction of M magnons with arbitrary quasimomenta the extended ansatz

$$\psi_{\{n_\alpha\}}=\sum_{P\in\pi_M}\Phi^{(0)}(P;\{n_\alpha\})\left[1+\sum_{L=3}^{M}\Omega_L\right] \tag{12}$$

where each term Ω_L vanishes if all $|n_\alpha-n_\beta|$ tend to infinity and can be represented as the sum Σ'_L over L indices varying from l to M so that their values do not coincide. The only requirement which determines the explicit form of Ω_L consists in the following: the contribution of Ω_L to the left-hand side of (2) must cancel in it all terms containing sums of the type Σ'_L which arise from the previous $\{\Omega_l\}$ with $l<L$. For example, the straightforward calculation with the use of (9) at $J=1$ and identity (10) shows that the contribution of $\psi^{(0)}_{\{n_\alpha\}}$ (11), which is of the type Σ'_3, is exactly cancelled if one chooses Ω_3 as

$$\Omega_3=\frac{1}{12}\sum_{\mu\neq\nu\neq\rho}^{M}\frac{C_{\mu\nu}^{(P)}C_{\mu\rho}^{(P)}C_{\nu\rho}^{(P)}}{S_{\mu\nu}^{(P)}S_{\mu\rho}^{(P)}S_{\nu\rho}^{(P)}} \tag{13}$$

but various sums of the structure $\Sigma'_4,\Sigma'_5,\Sigma'_6$ appear. The following term of the expansion (12) is determined so as to cancel the contribution from (13) to (2) of the Σ'_4 type,

$$\Omega_4=\frac{1}{32}\sum_{\mu\neq\nu\neq\rho\neq\lambda}^{M}\frac{C_{\mu\nu}^{(P)}C_{\mu\lambda}^{(P)}C_{\rho\nu}^{(P)}C_{\rho\lambda}^{(P)}}{S_{\mu\nu}^{(P)}S_{\mu\lambda}^{(P)}S_{\rho\nu}^{(P)}S_{\rho\lambda}^{(P)}}\left\{1-\sinh\left[\frac{\pi}{\kappa}(n_\mu-n_\rho)+\gamma(k_{P_\nu},k_{P_\lambda})\right]\right.$$
$$\left.\times\sinh\left[\frac{\hbar}{\kappa}(n_\nu-n_\lambda)-\gamma(k_{P_\mu},k_{P_\rho})\right](S_{\mu\rho}^{(P)}S_{\nu\lambda}^{(P)})^{-1}\right\}. \tag{14}$$

The complexity of the Ω_L construction essentially grows with L. For $L\geqslant 5$ we can formulate the following general procedure. Let us take on a plane L points supplied by summation indices $\{\mu\}$ with non-equal values. Connecting all these points by full lines so that an even number of them intersect at each point, we obtain a set of graphs $\{G\}$, as in figure 1 for $L=5$. Let us draw on each graph G also $m(G)$ broken lines so

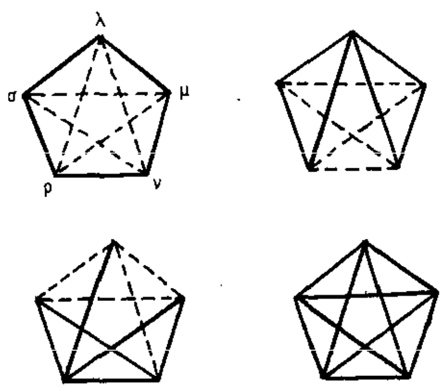

Figure 1. Graphs representing the structure of Ω_L with $L=5$. Due to the evenness of the numbers of full lines intersecting at each point, the calculations of sums in (2) containing Ω_L can be performed by the use of (9).

that finally each pair of points must be connected by a line of any type, and denote as F_G the variety of full lines in G. To each line $f_{\alpha\beta} \in F_G$ connecting two points with indices α and β we put into correspondence the factor

$$Q_f = \sinh \gamma(k_{P\alpha}, k_{P\beta}) \left\{ \sinh\left[\frac{\pi}{\kappa}(n_\alpha - n_\beta) + \gamma(k_{P\alpha}, k_{P\beta}) \right] \right\}^{-1}$$

that falls off exponentially with growing $|n_\alpha - n_\beta|$. For each pair of broken lines (b_l, b_n) connecting the points with indices (α, β) and (α', β'), $1 \leq l, n \leq m(G)$, let us construct the factor

$$\tilde{Q}_\delta(l, n) = \sinh\left[\frac{\pi}{\kappa}(n_\alpha - n_\beta) + \delta(l, n)\gamma(k_{P\alpha'}, k_{P\beta'}) \right] \times \left\{ \sinh\left[\frac{\pi}{\kappa}(n_\alpha - n_\beta) + \gamma(k_{P\alpha}, k_{P\beta}) \right] \right\}^{-1}$$

where $\delta(l, l) = 1$ and $\delta(l, n)$ may be equal to ± 1. The proper structure of Ω_L is now given by

$$\Omega_L = \sum_{\{\mu\}}{}'_L \sum_G \left(\prod_{f \in F_G} Q_f \right) \sum_{\tilde{P} \in \pi_{m(G)}} \sum_{\{\delta\}} A_L(G, \tilde{P}, \{\delta\}) \prod_{l=1}^{m(G)} \tilde{Q}_\delta(l, \tilde{P}l) \tag{15}$$

where $A_L(G, \tilde{P}, \{\delta\})$ are some numerical factors. They do not depend on κ and magnon quasimomenta and must be determined from the general definition of Ω_L. The examples of the construction (15) are given by the expressions (13) and (14) for $\Omega_{3,4}$. They are relatively simple because in both these cases there is only one graph of the type described above.

The number of graphs grows sharply with L; it is equal to 4 for $L = 5$ (figure 1), but already at $L = 6$ there are nine graphs with the numbers of full lines from 6 to 12. An effective algorithm for the calculation of $A_L(G; \tilde{P}, \{\delta\})$ for $L \geq 5$ has not yet been found.

At $M \leq 4$ the relations (12)-(14) give the explicit solution of the problem of interacting magnons in our model. The multimagnon scattering matrix is determined only by the two-magnon phase shift (6). In the limit $\kappa \to 0$

$$\coth \gamma(k_1, k_2) \to \frac{i}{2}\left(\cot\frac{k_1}{2} - \cot\frac{k_2}{2} \right)$$

i.e. one obtains just the expression for the Bethe phase. All Ω_L vanish in that limit because $S_{\mu\nu}^{(P)}$ tend to infinity and $C_{\mu\nu}^{(P)}$ remain finite. So (12) reduces to the Bethe ansatz. For the complex quasimomenta there are multimagnon bound states as in the infinite ferromagnetic Heisenberg chain. But, contrary to this, the connections between the real total quasimomentum $K = \sum_{\alpha=1}^{M} k_\alpha$ and complex relative ones must be obtained through the solution of the system of highly transcendental equations which include elliptic ζ functions

$$\coth \gamma(k_\alpha, k_{\alpha+1}) = 1 \qquad 1 \leq \alpha \leq M - 1. \tag{16}$$

The ground-state wavefunction for given K at the conditions (16) looks simpler than (12) and resembles the Jastrow structures, as it can be seen from its explicit form in the simplest non-trivial case $M = 3$

$$\psi^{(g)}_{n_1 n_2 n_3} = (s_{32}s_{31}s_{21})^{-1} \sum_{P \in \pi_3} \exp\left[i\left(\sum_{\alpha=1}^{3} k_\alpha n_{P\alpha} \right) + \frac{\pi}{\kappa}(n_{P1} - n_{P3}) \right] (-1)^P \tag{17}$$

where $(-1)^P$ is the parity of permutation P. To our knowledge, (17) is the first example which shows that the wavefunctions in the Jastrow form may be connected with the Bethe ansatz.

We conclude that, compared with the usual Heisenberg XXX chain, the model (1) permits the exact investigation of multi-magnon states in a more realistic case of non-nearest-neighbour (but short-range) spin interaction. One can expect that similar results may also be derived for the periodic version of the model by the use of more complicated calculations with elliptic functions.

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